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Dynamic of the singularly perturbed Nicholson equation

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Abstract:

In this paper, we consider a class of singularly perturbed equations with a parameter ϵ . By letting the perturbation parameter tends to zero, such an equation is formally reduced to a scalar difference equation. Local stability analysis of fixed points is investigated. The method of steps is used to discretize the system. Moreover, Numerical simulations including Lyapunov exponent, bifurcation and chaos is carried out to confirm the theoretical analysis obtained to explore more complex dynamic of the system.

Keywords: Fixed points, Local stability, Lyapunov exponent, Bifurcation and Chaos.

1 Introduction

Delay differential equation (DDE) are employed in modeling many problems in science, engineering, biology and medicine. That is because the delay terms in such equations increases the reliability in modeling real phenomena and makes the prediction of long-term behavior of such models more accurate.

Indeed, (DDE) represent dynamical system of infinite dimensions which is opposite to ordinary differential equations. Delay-differential equations are a big and important group of dynamical systems. The original stimulus for the study of delay differential equations mainly lies in the application of control theory and the study of stability and automated management. And its importance of controlling the time now in the formations and the wide range of applications.

Delays are ubiquitous in biology, arising from maturation, transcription, incubation and nerve impulse transmission time, to name but a few situations. Classical examples of delay equations in mathematical biology include the Mackey-Glass equation, Nicholson's blowflies equation [5], [9], and the delayed logistic equation, also known as Wright's equation, after a change of variable. Such equations have inspired decades of mathematical research into delay differential equation (DDEs), and there is now a well-established mathematical framework for problems with fixed or prescribed delays as infinite-dimensional dynamical systems on function spaces see [14], [10] for an easier treatment. Singular perturbation theory concerns the study of problems featuring a parameter for which the solutions of the problem at a limiting value of the parameter are different in

character from the limit of the solutions of the general problem, namely, the limit is singular. In contrast, for regular perturbation problems, the solutions of the general problem converge to the solutions of the limit-problem as the parameter approaches the limit-value[1].

The problem with classical singular perturbation theory [3], [2] lies in the fact that its results are expressed in a rather abstract and sophisticated way.

In the absence of spatial heterogeneity, many model equations from biology and other areas are of the form of the following delay differential equation:

$$\frac{dx}{dy} = -\delta x(t) + f(x(t - \tau)). \quad (1.1)$$

Here, the time delay $\tau > 0$ a prototype of such equations is the well known delayed Nicholson blowy equation (Nicholson [4] ; Gurney et al.[2])

$$\frac{dx}{dy} = -\delta x(t) + \rho x(t - \tau)e^{-ax(t-\tau)}, \quad t \in (0, T] \quad (1.2)$$

$$x(0) = x_0. \quad (1.3)$$

where δ, ρ, a are positive constant, $x(t)$ is the adult population of the y and $\tau > 0$ explains the maturation time of the fly. Equation (1.2) has been very well studied in [4], [5], [6] and in many research.

For Eq. (1.2), the following hold.

- If $\delta < \rho \leq \delta e^2$, then $x = x^*$ is asymptotically stable.
- If $\rho > \delta e^2$, then $x = x^*$ is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.
- For $\rho > \delta e^2$, Eq. (1.2) undergoes a Hopf bifurcation at x^* when $\tau > \tau_k$, for $k = 0, 1, 2, \dots$

2 Singularly perturbed systems

Consider the Nicholson delay differential equation

$$\epsilon \frac{dx}{dy} = -\delta x(t) + \rho x(t - \tau) e^{-ax(t-\tau)}, \quad (2.1)$$

$$x(\tau) = \phi(\tau), \quad \tau \leq 0. \quad (2.2)$$

For $\epsilon > 0$ small enough, we investigate the stability of the equilibrium points of the equation

$$\frac{dx}{dy} = \frac{-\delta}{\epsilon} x(t) + \frac{\rho}{\epsilon} x(t - \tau) e^{-ax(t-\tau)} \quad (2.3)$$

the equilibrium points are $x = 0$ and $x = 1$, if and only if $a > 0$ and $\rho > \delta$. There is a positive equilibrium

$$x^* = \frac{1}{a} \ln \left(\frac{\rho}{\delta} \right)$$

if the maximum possible per capita reproduction rate ρ is greater than the per capita death rate δ . As in the Nicholson equation, there is a critical value of the time delay. The positive equilibrium is stable when the delay is less the critical value, becomes unstable when it is greater the value, and there are oscillations.

To study the stability of the equilibrium points, we have

$$\begin{aligned} \hat{f}(x(t)) &= \left(\frac{-\delta}{\epsilon} \right) - \left(\frac{\rho}{\epsilon} \right) e^{-ax} (ax - 1). \\ \Rightarrow \hat{f}(0) &= \frac{\rho - \delta}{\epsilon}, \quad \hat{f}\left(\frac{1}{a} \ln \left(\frac{\rho}{\delta} \right)\right) = \frac{-\delta}{\epsilon} - \left(\frac{-\delta}{\epsilon} \right) \left(\ln \left(\frac{\rho}{\delta} \right) - 1 \right) \end{aligned}$$

It is known that the stability of equilibrium x^* points for equation (2.3) is determined by the characteristic equation after linearizing the equation (2.3) for the nontrivial equilibrium x^* .

$$\lambda + \left(\frac{\delta}{\epsilon} \right) \left(\ln \left(\frac{\rho}{\delta} \right) \right) e^{-\lambda \tau} = 0 \quad (2.4)$$

By the result of Hayes [13] (also see the appendix of Hale §Verduyn Lunel [1]), one knows that if $\hat{f}(x^*) > 1$, then equation (2.4) has a positive real root; and if $\hat{f}(x^*) < 1$, then

equation (2.4) will have complex roots with positive real parts when $\left(\frac{\delta}{\epsilon}\right)$ is large. That is, if $|\hat{f}(x^*)| > 1$, then x^* cannot be delay-independently asymptotically stable.

When $\rho > \delta e^2$. Equation (2.4) has of simple imaginary roots at $\tau > \tau_k$, $k = 0, 1, 2, \dots$

Then the equilibrium $x^* = \frac{1}{a} \ln\left(\frac{\rho}{\delta}\right)$ is asymptotically stable for $\tau \in [0, \tau_0)$ we find

$$\frac{-\delta}{\epsilon} \ln\left(\frac{\rho}{\delta}\right) < 1 \Rightarrow \rho > \delta e^{\frac{-\epsilon}{\delta}}, \text{ and when } \epsilon \rightarrow 0 \Rightarrow \rho > \delta.$$

3 The discretized system

In this section the discretized analogue of the system (2.3) is obtained via the method of steps [12].

By applying the method of steps then the equation

$$\frac{dx}{dy} = \frac{-\delta}{\epsilon} x(t) + \frac{\rho}{\epsilon} x(t - \tau) e^{-ax(t-\tau)}$$

have the solution as follows.

Let $t \in [0, \tau]$, then

$$\begin{aligned} x_1(t) &= e^{\frac{-\delta}{\epsilon}t} x_0 + \frac{1}{\epsilon} \int_0^t e^{\frac{-\delta}{\epsilon}(t-s)} f(x_0) ds \\ &= e^{\frac{-\delta}{\epsilon}t} x_0 + \rho x_0 e^{-ax_0} \left(\frac{1 - e^{\frac{-\delta}{\epsilon}(t)}}{\delta} \right), \end{aligned}$$

$(t \rightarrow \tau)$, then

$$x_1(\tau) = e^{\frac{-\delta}{\epsilon}\tau} x_0 + \rho x_0 e^{-ax_0} \left(\frac{1 - e^{\frac{-\delta}{\epsilon}(\tau)}}{\delta} \right)$$

For $t \in [\tau, 2\tau]$, we have

$$x_2(t) = e^{\frac{-\delta}{\epsilon}t} x_1 + \frac{1}{\epsilon} \int_{\tau}^t e^{\frac{-\delta}{\epsilon}(t-s)} f(x_1) ds,$$

$$= e^{\frac{-\delta}{\epsilon}t} x_1 + \rho x_1 e^{-ax_1} \left(\frac{1 - e^{\frac{-\delta}{\epsilon}(t)}}{\delta} \right),$$

$(t \rightarrow 2\tau)$

$$x_2(\tau) = e^{\frac{-\delta}{\epsilon}\tau} x_1 + \rho x_1 e^{-ax_1} \left(\frac{1 - e^{\frac{-\delta}{\epsilon}(\tau)}}{\delta} \right)$$

Repeating the process we can easily deduce that the solution of (1.2)-(2.2) is given by

$$x_{n+1}(t) = e^{\frac{-\delta}{\epsilon}t} x_n(t) + \frac{1}{\epsilon} \int_{n\tau}^t e^{\frac{-\delta}{\epsilon}(t-s)} f(x_n(t-s)) ds, \quad t \in [n\tau, (n+1)\tau].$$

By letting $t \rightarrow (n+1)\tau$, we obtain

$$x_{n+1}(\tau) = e^{\frac{-\delta}{\epsilon}n\tau} x_{n\tau} + \rho x_{n\tau} e^{-ax_{n\tau}} \left(\frac{1 - e^{\frac{-\delta}{\epsilon}(\tau)}}{\delta} \right). \quad (3.1)$$

3.1 Local Stability of equilibrium points

The system (3.1) -(2.2) has two fixed points, namely

$$(x_{fix})_1 = (0,0),$$

$$(x_{fix})_2 = \left(\frac{1}{a} \ln \left(\frac{\rho}{\delta} \right), \frac{1}{a} \ln \left(\frac{\rho}{\delta} \right) \right)$$

To study the local stability of these fixed points we rely on the following theorem.

Theorem 1. [11] Let f be a smooth map on R , and assume that x_0 is a fixed point of f .

1. If $|\dot{f}(x_0)| < 1$, then x_0 is stable.
2. If $|\dot{f}(x_0)| > 1$, then x_0 is unstable.

Now since our case $f(x) = e^{-\frac{\delta}{\epsilon}} + \frac{\rho(1 - e^{-\frac{\delta}{\epsilon}})}{\delta} e^{-ax}(1 - ax)$, the first fixed point $(x_{fixed})_1 = (0,0)$ is stable if

$$\left| \frac{e^{-\frac{\delta}{\epsilon}} + \rho(1 - e^{-\frac{\delta}{\epsilon}})}{\delta} \right| < 1,$$

that is, $\rho < \delta$.

The second fixed point $(x_{fixed})_2 = (\frac{1}{a} \ln(\frac{\rho}{\delta}), \frac{1}{a} \ln(\frac{\rho}{\delta}))$ is stable if

$$\rho - \rho \ln \frac{\rho}{\delta} < \delta.$$

4 Nicholsons map in case $\epsilon \rightarrow 0$

In this section consider the problem of the singularly perturbed differential equation (2.1) and (2.2). The limiting cases as $\epsilon \rightarrow 0$ of are special cases of the retarded functional equation

$$x(t) = \frac{\rho}{\delta} x(t - \tau) e^{-ax(t-\tau)}, \quad t \in (0, T], \quad (4.1)$$

with the initial condition (2.2).

Now apply the method of steps to the system of equations (4.1) and (2.2).

Step 1:

Let $t \in [0, \tau]$, then $t - \tau \in [-\tau, 0]$, the solution of (4.1)-(2.2) is given by

$$x(t) = x_1(\tau) = \frac{\rho}{\delta} x_0 e^{-ax_0}, \quad t \in [0, \tau].$$

Step 2:

For $t \in [\tau, 2\tau]$, then $t - \tau \in [0, \tau]$, the solution of (4.1)-(2.2) is given by

$$x(t) = x_2(\tau) = f(x_1(t)) = \frac{\rho}{\delta} x_1 e^{-ax_1}, \quad t \in [\tau, 2\tau].$$

Step n:

Repeating the process we can easily deduce that the solution of (4.1)-(2.2) is given by

$$x_{n+1} = \frac{\rho}{\delta} x_n e^{-ax_n}, \quad t \in [n\tau, (n+1)\tau]. \quad (4.2)$$

4.1 Fixed points and stability

Exactly like its discrete counter part, the dynamical system (4.2)-(2.2) has two fixed points which are the solution of the equation

$$x = \frac{\rho}{\delta} x e^{-ax}.$$

So, we have

$$\begin{aligned} (x_{fix})_1 &= (0, 0) \\ (x_{fix})_2 &= \left(\frac{1}{a} \ln \left(\frac{\rho}{\delta} \right), \frac{1}{a} \ln \left(\frac{\rho}{\delta} \right) \right) \end{aligned}$$

Now since in our case $f(x) = \frac{\rho}{\delta} x e^{-ax}$, the first fixed point $(x_{fixed})_1 = (0, 0)$ is stable if

$$\left| \frac{\rho}{\delta} \right| < 1,$$

that is, $\rho < \delta$.

The second fixed point $(x_{fixed})_2 = \left(\frac{1}{a} \ln \left(\frac{\rho}{\delta} \right), \frac{1}{a} \ln \left(\frac{\rho}{\delta} \right) \right)$ is stable if

$$\left| 1 - \ln \frac{\rho}{\delta} \right| < 1.$$

So, the second fixed point is stable if $\rho > \delta$.

4.2 Numerical results

In this section, the numerical experiments show that the dynamical behaviors of dynamical system (4.1), since the Lyapunov exponent is a good indicator for existence of chaos, we compute the Lyapunov Characteristic Exponents (LCEs) via the Householder QR Based Methods described in [15]. LCEs play a key role in the study of nonlinear dynamical systems and they are a measure of the sensitivity of the solutions of a given dynamical system to small changes in the initial conditions. One feature of chaos is the sensitive dependence on initial conditions; for a chaotic dynamical system at least one LCE must be positive. Since for non-chaotic systems all LCEs are non-positive, the presence of a positive LCE has often been used to help determine if a system is chaotic or not.

Figure (1) shows the LCEs and bifurcation diagram for system (3.1)-(2.2) we see clearly in this figure the bifurcation from a stable equilibrium point to a stable orbit of period 2 at $\rho = 15$, and then the bifurcation from period two to period four at ρ between 25 and 30. The further period doubling occur at decreasing increments in ρ , and the orbit becomes chaotic for $\rho \approx 30$. Note the intriguing window just beyond 45.

The same can be said to figure (2) shows the LCEs and bifurcation diagram for system (4.2)-(2.2).

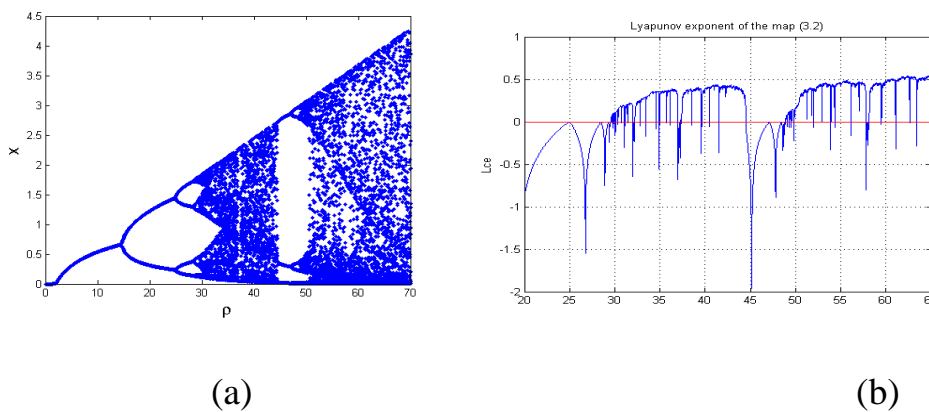


Figure 1: bifurcation diagram and Lyapunov exponent for system (3.1).

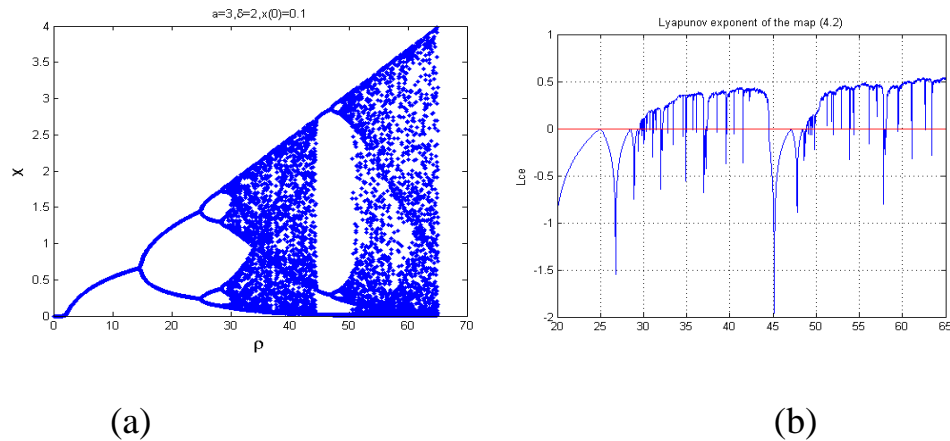


Figure 2: bifurcation diagram and Lyapunov exponent for system (4.2).

5 Conclusion

In this work we studied the dynamics local stability of delay differential Nicholson equation and applying the method of steps for problem of the singularly perturbed differential equation (2.1) and (2.2). Moreover, chaos and bifurcation of the resulting discrete system were numerically investigated by varying the system parameter ρ .



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